

# A FINITE PRESENTATION FOR THE AUTOMORPHISM GROUP OF THE FIRST HOMOLOGY OF A NON-ORIENTABLE SURFACE OVER $\mathbb{Z}_2$ PRESERVING THE MOD 2 INTERSECTION FORM

RYOMA KOBAYASHI AND GENKI OMORI

ABSTRACT. Let  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  be the group of automorphisms on the first homology group with  $\mathbb{Z}_2$  coefficients of a closed non-orientable surface  $N_g$  preserving the mod 2 intersection form. In this paper, we obtain a finite presentation for  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ . As an application we calculate the second homology group of  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ .

## 1. INTRODUCTION

For  $g \geq 1$  and  $n \geq 0$ , let  $N_{g,n}$  be a compact connected non-orientable surface of genus  $g$  with  $n$  boundary components (we denote  $N_{g,0}$  by  $N_g$ ) and a bilinear form  $\cdot : H_1(N_g; \mathbb{Z}_2) \times H_1(N_g; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$  the mod 2 intersection form on the first homology group  $H_1(N_g; \mathbb{Z}_2)$  of  $N_g$  with  $\mathbb{Z}_2$  coefficients. We represent  $N_g$  by a sphere with  $g$  crosscaps as in Figure 1, i.e. we regard  $N_g$  as a sphere with  $g$  boundary components attached a Möbius band to each boundary component. We define  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  by the subgroup of the automorphism group  $\text{Aut } H_1(N_g; \mathbb{Z}_2)$  of  $H_1(N_g; \mathbb{Z}_2)$  preserving the mod 2 intersection form  $\cdot$ . Note that  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  is isomorphic to  $O(g, \mathbb{Z}_2) = \{A \in GL(g, \mathbb{Z}_2) \mid {}^tAA = E\}$  by taking the basis  $\{x_1, x_2, \dots, x_g\}$  for  $H_1(N_g; \mathbb{Z}_2)$ , where  $x_i$  is a homology class of a one-sided simple closed curve  $\mu_i$  in Figure 1 and  $E$  is an identity matrix of  $GL(g, \mathbb{Z}_2)$  (cf. [5]). By Korkmaz [3] and Szepietowski [12] we have isomorphisms

$$\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot) \cong \begin{cases} \text{Sp}(2h, \mathbb{Z}_2) & \text{if } g = 2h + 1, \\ \text{Sp}(2h, \mathbb{Z}_2) \ltimes \mathbb{Z}_2^{2h+1} & \text{if } g = 2h + 2. \end{cases}$$

Let  $a_i$  ( $i = 1, \dots, g-1$ , for  $g \geq 2$ ),  $b$  (for  $g \geq 4$ )  $\in \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  be the following elements:

$$a_i : \begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_i, \\ x_k & \mapsto x_k \quad (k \neq i, i+1), \end{cases} \quad b : \begin{cases} x_1 & \mapsto x_2 + x_3 + x_4, \\ x_2 & \mapsto x_1 + x_3 + x_4, \\ x_3 & \mapsto x_1 + x_2 + x_4, \\ x_4 & \mapsto x_1 + x_2 + x_3, \\ x_k & \mapsto x_k \quad (k \neq 1, 2, 3, 4). \end{cases}$$

In this paper, we give a finite presentation for  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ .

**Theorem 1.1.** *If  $g = 1, 2, 3$ , then  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  is the following group.*

- $\text{Aut}(H_1(N_1; \mathbb{Z}_2), \cdot) = 1$ ,
- $\text{Aut}(H_1(N_2; \mathbb{Z}_2), \cdot) = \langle a_1 \mid a_1^2 = 1 \rangle \cong \mathbb{Z}_2$ ,

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*Date:* April 7, 2015.

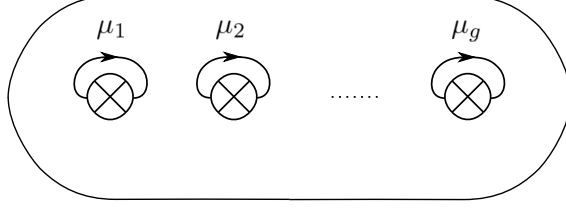


FIGURE 1. Simple closed curves  $\mu_1, \mu_2, \dots, \mu_g$  in  $N_g$  representing the basis  $x_1, x_2, \dots, x_g$  for  $H_1(N_g; \mathbb{Z}_2)$  respectively.

- $\text{Aut}(H_1(N_3; \mathbb{Z}_2), \cdot) = \langle a_1, a_2 \mid a_1^2 = a_2^2 = (a_1 a_2)^3 = 1 \rangle$ .

If  $g \geq 4$  is odd,  $g = 4$  or  $6$ , then  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  admits a presentation with generators  $a_1, \dots, a_{g-1}, b$  and relations:

- (1)  $a_i^2 = b^2 = 1$  for  $i = 1, \dots, g-1$ ,
- (2)  $(a_i a_j)^2 = 1$  for  $g \geq 4$ ,  $|i - j| > 1$ ,
- (3)  $(a_i a_{i+1})^3 = 1$  for  $g \geq 3$ ,  $i = 1, \dots, g-2$ ,
- (4)  $(a_i b)^2 = 1$  for  $g \geq 4$ ,  $i \neq 4$ ,
- (5)  $(a_4 b)^3 = 1$  for  $g \geq 5$ ,
- (6)  $(a_2 a_3 a_4 a_5 a_6 b)^{12} = (a_1 a_2 a_3 a_4 a_5 a_6 b)^9$  for  $g \geq 7$ .

If  $g \geq 8$  is even, then  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  admits a presentation with generators  $a_1, \dots, a_{g-1}, b, b_0, \dots, b_{\frac{g-2}{2}}$  and relations (1)-(6) above and the following relations:

- (7)  $b_0 = a_1$ ,  $b_1 = b$ ,  $b_2 = (a_1 a_2 a_3 a_4 a_5 b)^5$ ,
- (8)  $b_{i+1} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5 (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^{-6}$  for  $2 \leq i \leq \frac{g-4}{2}$ ,
- (9)  $[a_{g-5}, b_{\frac{g-2}{2}}] = 1$ .

We read every word of every group in this paper from right to left. In Section 3, we will prove Theorem 1.1 for  $g \geq 4$ . Theorem 1.1 is clear for  $g = 1, 2$ . For  $g = 3, 4$ , Szepietowski [11, in the proof of Theorem 5.5] gave the presentation for  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ . Note that  $\text{Aut}(H_1(N_3; \mathbb{Z}_2), \cdot)$  is isomorphic to the dihedral group  $D_6$  and the symmetric group  $S_3$ . By the result of Korkmaz [3, Corollary 4.1] and Theorem 1.1, the first homology group of  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  is as follows.

$$H_1(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z}) = \begin{cases} 0 & \text{for } g = 1, g \geq 7, \\ \langle [a_1] \rangle \cong \mathbb{Z}_2 & \text{for } g = 2, 3, 5, 6, \\ \langle [a_1], [b] \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } g = 4. \end{cases}$$

Note that the above equality is known for  $g \geq 7$  odd (see, for instance, [13]).

In Section 4, by using the presentation for  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  obtained in Theorem 1.1, we calculate the second homology group of  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  for  $g \geq 9$ . We get the following theorem.

**Theorem 1.2.** *For  $g \geq 9$  or  $g = 7$ , the second homology group of  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  is trivial.*

Theorem 1.2 was shown by Stein [8] for odd  $g$  (see Theorem 2.13 and Proposition 3.3 (a)). More precisely, Stein proved  $H_2(\text{Sp}(2h, \mathbb{Z}_m); \mathbb{Z}) = 0$  when  $h \geq 3$  and  $m$  is not divisible by 4 (see also [1]).

To prove Theorem 1.2, we give a generating set for  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  which consists of one element  $x_0$  by an application of the discussion of Pitsch [7].

By using the generator of  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  and Stein's result we show that  $x_0$  is trivial for  $g \geq 9$ .

## 2. PRELIMINARIES

Let  $\alpha_1, \dots, \alpha_{g-1}, \beta$  be two-sided simple closed curves on  $N_g$  as in Figure 2. Arrows on the side of simple closed curves in Figure 2 indicate directions of Dehn twists along their simple closed curves. Since the actions of the Dehn twists along  $\alpha_1, \dots, \alpha_{g-1}, \beta$  induce  $a_1, \dots, a_{g-1}, b$  on  $H_1(N_g; \mathbb{Z}_2)$  respectively, we denote Dehn twists along  $\alpha_1, \dots, \alpha_{g-1}, \beta$  by  $a_1, \dots, a_{g-1}, b$  and abuse the notation.

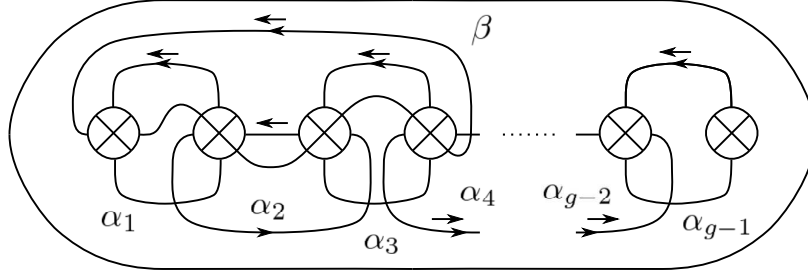


FIGURE 2. Simple closed curves  $\alpha_1, \dots, \alpha_{g-1}, \beta$  on  $N_g$ .

Let  $\mu$  be a one-sided simple closed curve and  $\alpha$  a two-sided simple closed curve such that  $\mu$  and  $\alpha$  intersect transversely in one point. For these simple closed curves  $\mu$  and  $\alpha$ , we denote by  $Y_{\mu, \alpha}$  a self-diffeomorphism on  $N_g$  which is described as the result of pushing the regular neighborhood of  $\mu$  once along  $\alpha$  (see Figure 3). We call  $Y_{\mu, \alpha}$  a *Y-homeomorphism*. We set the direction of  $Y_{\mu_i, \alpha_j}$  ( $1 \leq i \leq g$ ,  $1 \leq j \leq g-1$ ) by the orientation of  $\alpha_j$  in Figure 2 and  $y := Y_{\mu_1, \alpha_1}$ . Note that the action of Y-homeomorphism on  $H_1(N_g; \mathbb{Z}_2)$  is trivial.

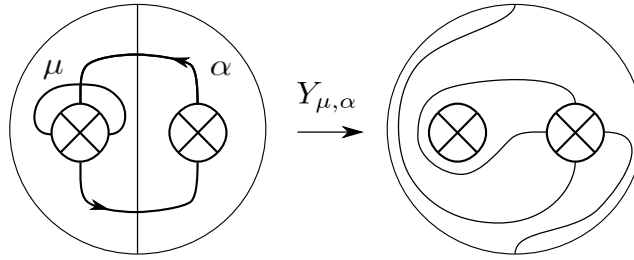


FIGURE 3. The Y-homeomorphism on the regular neighborhood of  $\mu \cup \alpha$ .

The *mapping class group*  $\mathcal{M}(N_{g,n})$  of  $N_{g,n}$  is the group of isotopy classes of self-diffeomorphisms on  $N_g$  fixing each boundary component pointwise. Paris and Szepietowski [6] gave a finite presentation for  $\mathcal{M}(N_g)$ . The presentation has a generating set which consists of Dehn twists along two-sided simple closed curves and “crosscap transpositions”. Stukow [10] obtained a finite presentation for  $\mathcal{M}(N_g)$  whose generators are Dehn twists and a Y-homeomorphism. Stukow's presentation is the following.

**Theorem 2.1** ([10]). *If  $g \geq 4$  is odd or  $g = 4$ , then  $\mathcal{M}(N_g)$  admits a presentation with generators  $a_1, \dots, a_{g-1}, b, y$  and  $\rho$ . The defining relations are*

- (A1)  $[a_i, a_j] = 1$  for  $|i - j| > 1$ ,
- (A2)  $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$  for  $i = 1, \dots, g - 2$ ,
- (A3)  $[a_i, b] = 1$  for  $i \neq 4$ ,
- (A4)  $a_4 b a_4 = b a_4 b$  for  $g \geq 5$ ,
- (A5)  $(a_2 a_3 a_4 b)^{10} = (a_1 a_2 a_3 a_4 b)^6$  for  $g \geq 5$ ,
- (A6)  $(a_2 a_3 a_4 a_5 a_6 b)^{12} = (a_1 a_2 a_3 a_4 a_5 a_6 b)^9$  for  $g \geq 7$ ,
- (B1)  $y(a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1}) = (a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1})y$ ,
- (B2)  $y(a_2 a_1 y^{-1} a_2^{-1} y a_1 a_2)y = a_1(a_2 a_1 y^{-1} a_2^{-1} y a_1 a_2)a_1$ ,
- (B3)  $[a_i, y] = 1$  for  $i = 3, \dots, g - 1$ ,
- (B4)  $a_2(y a_2 y^{-1}) = (y a_2 y^{-1})a_2$ ,
- (B5)  $y a_1 = a_1^{-1} y$ ,
- (B6)  $b y b y^{-1} = \{a_1 a_2 a_3 (y^{-1} a_2 y) a_3^{-1} a_2^{-1} a_1^{-1}\} \{a_2^{-1} a_3^{-1} (y a_2 y^{-1}) a_3 a_2\}$ ,
- (B7)  $[(a_4 a_5 a_3 a_4 a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1} a_4^{-1} a_3^{-1} a_5^{-1} a_4^{-1}), b] = 1$  for  $g \geq 6$ ,
- (B8)  $\{(y a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1}) b (a_4 a_3 a_2 a_1 y^{-1})\} \{(a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1}) b^{-1} (a_4 a_3 a_2 a_1)\}$   
 $= \{(a_4^{-1} a_3^{-1} a_2^{-1}) y (a_2 a_3 a_4)\} \{a_3^{-1} a_2^{-1} y^{-1} a_2 a_3\} \{a_2^{-1} y a_2\} y^{-1}$  for  $g \geq 5$ ,
- (C1a)  $(a_1 a_2 \cdots a_{g-1})^g = \rho$  for  $g$  odd,
- (C1b)  $(a_1 a_2 \cdots a_{g-1})^g = 1$  for  $g$  even,
- (C2)  $[a_1, \rho] = 1$ ,
- (C3)  $\rho^2 = 1$ ,
- (C4a)  $(y^{-1} a_2 a_3 \cdots a_{g-1} y a_2 a_3 \cdots a_{g-1})^{\frac{g-1}{2}} = 1$  for  $g$  odd,
- (C4b)  $(y^{-1} a_2 a_3 \cdots a_{g-1} y a_2 a_3 \cdots a_{g-1})^{\frac{g-2}{2}} y^{-1} a_2 a_3 \cdots a_{g-1} = \rho$  for  $g$  even,

where  $[X, Y] = XYX^{-1}Y^{-1}$ . If  $g \geq 6$  is even then  $\mathcal{M}(N_g)$  admits a presentation with generators  $a_1, \dots, a_{g-1}, y, b, \rho$  and  $b_0, \dots, b_{\frac{g-2}{2}}$ . The defining relations are (A1)-(A6), (B1)-(B8), (C1a)-(C4b) above and the following relations:

- (A7)  $b_0 = a_1, b_1 = b$ ,
- (A8)  $b_{i+1} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5 (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^{-6}$   
for  $1 \leq i \leq \frac{g-4}{2}$ ,
- (A9a)  $[b_2, b] = 1$  for  $g = 6$ ,
- (A9b)  $[a_{g-5}, b_{\frac{g-2}{2}}] = 1$  for  $g \geq 8$ .

Relations (A1) and (A3) are called *disjointness relations* and relations (A2) and (A4) are called *braid relations*. When we deform relations (or words) by disjointness relations and braid relations, we write “DI” and “BR” on the left-right arrow (or the equality sign) respectively.

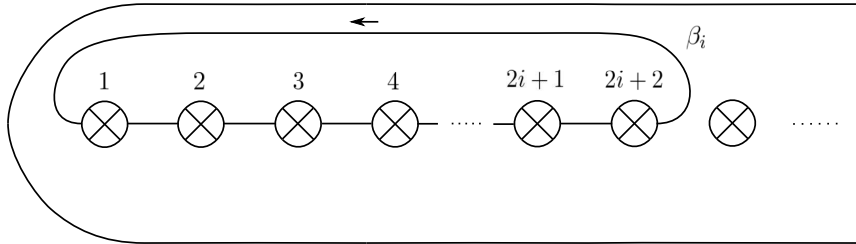


FIGURE 4. Simple closed curves  $\beta_i$  on  $N_g$  for  $2 \leq i \leq \frac{g-2}{2}$ .

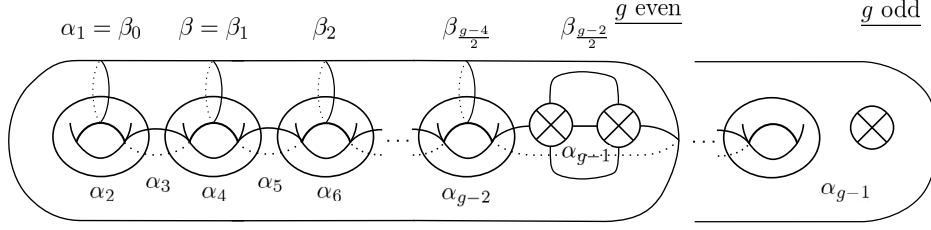


FIGURE 5. A different view of simple closed curves  $\alpha_i$  ( $1 \leq i \leq g-1$ ) and  $\beta_j$  ( $0 \leq j \leq \frac{g-2}{2}$ ) on  $N_g$ .

$b_i$  ( $2 \leq i \leq \frac{g-2}{2}$ ) in the  $g$  even case of Theorem 2.1 is the Dehn twist along a simple closed curve  $\beta_i$  in Figure 4. The arrow on the side of the simple closed curve  $\beta_i$  in Figure 4 indicates the direction of the Dehn twist  $b_i$ . We note that  $N_g$  is diffeomorphic to a surface as in Figure 5 and we can choose the diffeomorphism such that simple closed curves  $\alpha_i$  ( $1 \leq i \leq g-1$ ) in Figure 2 and  $\beta_j$  ( $0 \leq j \leq \frac{g-2}{2}$ ) in Figure 4 are sent to a position in Figure 5.

Since the action of  $\mathcal{M}(N_g)$  on  $H_1(N_g; \mathbb{Z}_2)$  preserves the mod 2 intersection form  $\cdot$ , we have a homomorphism  $\rho_2 : \mathcal{M}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ . McCarthy and Pinkall [5] showed that  $\rho_2$  is surjective.  $\Gamma_2(N_g) := \ker \rho_2$  is called the *level 2 mapping class group* of  $\mathcal{M}(N_g)$ . Szepietowski [11] proved that  $\Gamma_2(N_g)$  is generated by Y-homeomorphisms for  $g \geq 2$ . More precisely, Szepietowski showed the following theorem.

**Theorem 2.2** ([11]). *For  $g \geq 2$ ,  $\Gamma_2(N_g)$  is normally generated by  $y$  in  $\mathcal{M}(N_g)$ .*

We note that squares of Dehn twists along non-separating two-sided simple closed curves are elements of  $\Gamma_2(N_g)$ . Hence  $\{a_1^2, \dots, a_{g-1}^2, b^2, y\}$  is a normal generating set for  $\Gamma_2(N_g)$  in  $\mathcal{M}(N_g)$ .

We now explain about the Tietze transformations. Let  $G$  be a group with presentation  $G = \langle X | R \rangle$ , where  $X$  is a subset of  $G$  and  $R$  is a set consisting of words of elements of  $X$ . Then  $G$  is isomorphic to the quotient group  $F/K$ , where  $F$  is the free group which is generated by  $X$  and  $K$  is the normal subgroup of  $F$  which is normally generated by  $R$ . Then the following transformations among presentations do not change the isomorphism class of  $G$ .

$$\begin{aligned} \langle X | R \rangle &\longleftrightarrow \langle X | R \cup \{k\} \rangle && \text{for } k \in K - R, \\ &\longleftrightarrow \langle X \cup \{v\} | R \cup \{vw^{-1}\} \rangle && \text{for } w \in F - X. \end{aligned}$$

These transformations are called the *Tietze transformations*. In this paper, we use these transformations without any comment when we deform presentations (or relations).

### 3. PROOF OF THEOREM 1.1 FOR $g \geq 4$

By the definition of  $\Gamma_2(N_g)$  and surjectivity of  $\rho_2 : \mathcal{M}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ , we have the following short exact sequence.

$$(3.1) \quad 1 \longrightarrow \Gamma_2(N_g) \longrightarrow \mathcal{M}(N_g) \xrightarrow{\rho_2} \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot) \longrightarrow 1.$$

We have the finite presentation for  $\mathcal{M}(N_g)$  (Theorem 2.1) and the normal generating set  $\{a_1^2, \dots, a_{g-1}^2, b^2, y\}$  for  $\Gamma_2(N_g)$  (Theorem 2.2). We can get a presentation for  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  by adding  $\{a_1^2, \dots, a_{g-1}^2, b^2, y\}$  to the relations of the presentation for  $\mathcal{M}(N_g)$  in Theorem 2.1.

The relations  $a_1^2 = \dots = a_{g-1}^2 = b^2 = 1$  are nothing but relations (1) in Theorem 1.1 clearly. By Claim 3.2 and relations (1), we have

$$\begin{aligned} (a_1 a_2 a_3 a_4 a_5)^6 &= (a_1^2 a_2 a_3 a_4 a_5)^5 \\ &= (a_2 a_3 a_4 a_5)^5 \\ &= (a_2^2 a_3 a_4 a_5)^4 \\ &\vdots \\ &= a_5^2 \\ &= 1. \end{aligned}$$

Hence we obtain the relation  $b_2 = (a_1 a_2 a_3 a_4 a_5 b)^5$  in relation (7) from relation (A8) for  $i = 1$ . Relations (2), (3), (4), (5) and (9) in Theorem 1.1 are obtained from relations (A1), (A2), (A3), (A4) and (A9b) in Theorem 2.1, and relations (1).

Since  $y = 1$  in  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ , relations (B1), (B3), (B4), (B7), (B8) are unnecessary. By using relations (1) and braid relations (relations (A2), (A4) in Theorem 2.1), relations (B2), (B5), (B6) are deformed as follows.

$$\begin{aligned} \text{(B2)} \quad & \xleftrightarrow{y=1 \& (1)} a_2 a_1 a_2 a_1 a_2 = \underline{a_1 a_2 a_1 a_2 a_1 a_2 a_1} \xleftrightarrow{\text{BR}} a_2 a_1 a_2 a_1 a_2 = a_2 a_1 \underline{a_2 a_2 a_2 a_1 a_2} \\ & \xleftrightarrow{(1)} a_2 a_1 a_2 a_1 a_2 = a_2 a_1 a_2 a_1 a_2. \\ \text{(B5)} \quad & \xleftrightarrow{y=1} a_1 = a_1^{-1} \iff (1). \\ \text{(B6)} \quad & \xleftrightarrow{y=1 \& (1)} 1 = a_1 \underline{a_2 a_3 a_2 a_3 a_2 a_1 a_2 a_3 a_2} \xleftrightarrow{\text{BR}} 1 = a_1 a_3 \underline{a_2 a_3 a_3 a_2 a_1 a_2 a_3 a_3 a_2 a_3} \\ & \iff 1 = a_1 a_3 a_1 a_3 \xleftrightarrow{(A1) \& (1)} 1 = a_1^2. \end{aligned}$$

Therefore relations (B1), (B2),  $\dots$ , (B8) drop out.

It is sufficient for proof of this theorem to show the following three claims:

**Claim 3.1.** *Relations (C1a), (C4b) are equivalent to  $\rho = 1$  under  $y = 1$ , relations (1), (BR) and (DI). It allows you to rule out generator  $\rho$  and relations (C1a), (C2), (C3) and (C4b) from the presentation.*

**Claim 3.2.** *Let  $G$  be a group and assume that  $g_1, g_2, \dots, g_n \in G$  satisfy relations*

$$\begin{aligned} \text{(BR)} \quad & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for } i = 1, \dots, n-1, \\ \text{(DI)} \quad & [g_i, g_j] = 1 \quad \text{for } |i - j| > 1. \end{aligned}$$

*Then we have a relation  $(g_1 g_2 \dots g_n)^{n+1} = (g_1^2 g_2 \dots g_n)^n$  on  $G$ .*

*“BR” and “DI” means braid relations and disjointness relations, respectively.*

**Claim 3.3.** *Relation (A9a) follows from relations (1), (2), (3), (4), (5).*

We suppose that Claim 3.2 and Claim 3.3 are true.

**Proof of Claim 3.1.** Since  $y = 1$  in  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  and  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  has relations (C1a) and (C4b),  $\rho$  is represented by the form

$$\rho = \begin{cases} (a_1 a_2 \dots a_{g-1})^g & \text{for } g \text{ odd,} \\ (a_2 \dots a_{g-1})^{g-1} & \text{for } g \text{ even.} \end{cases}$$

We get  $\rho = 1$  by repeatedly applying Claim 3.2 and relations (1) to the right-hand side of the above equation. For example, in  $g$  odd case:

$$\begin{aligned} \rho &= (a_1 a_2 \cdots a_{g-1})^g \stackrel{\text{Claim 3.2}}{=} (a_2 \cdots a_{g-1})^{g-1} \stackrel{\text{Claim 3.2}}{=} (a_3 \cdots a_{g-2})^{g-2} \\ &\stackrel{\text{Claim 3.2}}{=} \cdots \stackrel{\text{Claim 3.2}}{=} a_{g-1}^2 \stackrel{(1)}{=} 1. \end{aligned}$$

Thus we obtain the claim.  $\square$

By Claim 3.1, relations (C2) and (C3) are unnecessary. By a discussion similar to the proof of Claim 3.1, relations (C1b) and (C4a) are unnecessary, too. Therefore relations (C1a), (C1b), (C2), (C3), (C4a), (C4b) drop out. For relation (A5), we apply Claim 3.2 as follows.

$$\begin{aligned} (A5) &\iff (a_2 a_3 a_4 b)^{10} = \underline{(a_1 a_2 a_3 a_4 b)^6} \stackrel{\text{Claim 3.2}}{\iff} (a_2 a_3 a_4 b)^{10} = (a_2 a_3 a_4 b)^5 \\ &\iff (a_2 a_3 a_4 b)^5 = 1 \stackrel{\text{Claim 3.2}}{\iff} \cdots \stackrel{\text{Claim 3.2}}{\iff} b^2 = 1 \iff (1). \end{aligned}$$

We have completed the proof of Theorem 1.1 without proofs of Claim 3.2 and Claim 3.3.

### Proof of Claim 3.2.

$$(g_1 g_2 \cdots g_n)^{n+1} = (g_1 g_2 \cdots g_n)(g_1 g_2 \cdots g_n) \cdots (g_1 g_2 \cdots g_n)$$

Let  $A_i$  ( $i = n+1, n, \dots, 1$ ) be the  $i$ -th sequence  $(g_1 g_2 \cdots g_n)$  from the right in the right-hand side. By using disjointness relations and braid relations, the above equation is deformed as follows.

$$\begin{aligned} (g_1 g_2 \cdots g_n)^{n+1} &= A_{n+1} A_n \cdots A_1 \\ &= (g_1 g_2 \cdots g_{n-1} \underline{g_n}) A_n A_{n-1} \cdots A_1 \\ &\stackrel{\text{DI}}{=} (g_1 g_2 \cdots g_{n-1})(g_1 g_2 \cdots g_{n-2} \underline{g_n g_{n-1} g_n}) A_{n-1} \cdots A_1 \\ &\stackrel{\text{BR}}{=} (g_1 g_2 \cdots g_{n-1})(g_1 g_2 \cdots g_n) g_{n-1} A_{n-1} \cdots A_1. \end{aligned}$$

We replace the first sequence  $(g_1 g_2 \cdots g_n)$  from the left in the bottom with  $A_n$ . Then we have

$$\begin{aligned} (g_1 g_2 \cdots g_n)^{n+1} &= (g_1 g_2 \cdots g_{n-1}) A_n \underline{g_{n-1}} A_{n-1} \cdots A_1 \\ &\stackrel{\text{DI}}{=} (g_1 g_2 \cdots g_{n-1}) A_n (g_1 g_2 \cdots g_{n-3} \underline{g_{n-1} g_{n-2} g_{n-1} g_n}) A_{n-2} \cdots A_1 \\ &\stackrel{\text{BR}}{=} (g_1 g_2 \cdots g_{n-1}) A_n (g_1 g_2 \cdots g_{n-3} g_{n-2} g_{n-1} \underline{g_{n-2} g_n}) A_{n-2} \cdots A_1 \\ &\stackrel{\text{DI}}{=} (g_1 g_2 \cdots g_{n-1}) A_n (g_1 g_2 \cdots g_n) g_{n-2} A_{n-2} \cdots A_1. \end{aligned}$$

We replace the second sequence  $(g_1 g_2 \cdots g_n)$  from the left in the bottom with  $A_{n-1}$  and repeat it. Then we have

$$\begin{aligned}
(g_1 g_2 \cdots g_n)^{n+1} &= (g_1 g_2 \cdots g_{n-1}) A_n A_{n-1} g_{n-2} A_{n-2} \cdots A_1 \\
&= (g_1 g_2 \cdots g_{n-1}) A_n A_{n-1} A_{n-2} g_{n-3} A_{n-3} \cdots A_1 \\
&\vdots \\
&= (g_1 g_2 \cdots g_{n-1}) A_n \cdots A_2 g_1 A_1 \\
&= (g_1 g_2 \cdots g_{n-2}) A_n g_{n-2} A_{n-1} \cdots A_2 g_1 A_1 \\
&\vdots \\
&= (g_1 g_2 \cdots g_{n-2}) A_n \cdots A_3 g_1 A_2 g_1 A_1 \\
&\vdots \\
&= g_1 A_n \cdots g_1 A_3 g_1 A_2 g_1 A_1 \\
&= (g_1^2 g_2 \cdots g_n)^n.
\end{aligned}$$

Thus we obtain the claim.  $\square$

**Proof of Claim 3.3.** Note that  $b_2 = (a_1 a_2 a_3 a_4 a_5 b)^5$ . We first show the followings.

- (a)  $a_i(a_1 a_2 a_3 a_4 a_5 b) = (a_1 a_2 a_3 a_4 a_5 b) a_{i-1}$  for  $i = 2, 3, 4$ .
- (b)  $b(a_1 a_2 a_3 a_4 a_5 b) = (a_1 a_2 a_3 a_4 a_5 b) a_4 a_5 a_4$ .
- (c)  $a_5(a_1 a_2 a_3 a_4 a_5 b) = (a_1 a_2 a_3 a_4 a_5 b) a_4 b a_4$ .

Relation (a) is obtained by an argument similar to the proof of Claim 3.2. The other relations are obtained by the following deformations.

$$\begin{aligned}
\text{(b)} \quad b(a_1 a_2 a_3 a_4 a_5 b) &\stackrel{\text{DI}}{=} a_1 a_2 a_3 \underline{b a_4 b a_5} \stackrel{\text{BR}}{=} a_1 a_2 a_3 a_4 b a_4 a_5 \stackrel{(1)}{=} a_1 a_2 a_3 a_4 b (a_5 a_5) a_4 a_5 \\
&\stackrel{\text{BR}}{=} (a_1 a_2 a_3 a_4 a_5 b) a_4 a_5 a_4. \\
\text{(c)} \quad a_5(a_1 a_2 a_3 a_4 a_5 b) &\stackrel{\text{DI}}{=} a_1 a_2 a_3 \underline{a_5 a_4 a_5 b} \stackrel{\text{BR}}{=} a_1 a_2 a_3 a_4 a_5 a_4 b \stackrel{(1)}{=} a_1 a_2 a_3 a_4 a_5 (\underline{b b}) a_4 b \\
&\stackrel{\text{BR}}{=} (a_1 a_2 a_3 a_4 a_5 b) a_4 b a_4.
\end{aligned}$$

We now prove  $bb_2 = b_2 b$  by using only relations (a), (b), (c), (1) and disjointness relations. It means the relation  $bb_2 = b_2 b$  is unnecessary.

$$\begin{aligned}
bb_2 &= b(a_1 a_2 a_3 a_4 a_5 b)^5 \\
&\stackrel{(b)}{=} (a_1 a_2 a_3 a_4 a_5 b) a_4 a_5 a_4 (a_1 a_2 a_3 a_4 a_5 b)^4 \\
&\stackrel{(a),(c)}{=} (a_1 a_2 a_3 a_4 a_5 b)^2 a_3 a_4 b a_4 a_3 (a_1 a_2 a_3 a_4 a_5 b)^3 \\
&\stackrel{(a),(b)}{=} (a_1 a_2 a_3 a_4 a_5 b)^3 a_2 a_3 a_4 a_5 a_4 a_3 a_2 (a_1 a_2 a_3 a_4 a_5 b)^2 \\
&\stackrel{(a),(c)}{=} (a_1 a_2 a_3 a_4 a_5 b)^4 a_1 a_2 a_3 a_4 \underline{b a_4 a_3 a_2 a_1} (a_1 a_2 a_3 a_4 a_5 b) \\
&\stackrel{(1)}{=} (a_1 a_2 a_3 a_4 a_5 b)^4 a_1 a_2 a_3 a_4 \underline{b a_5 b} \\
&\stackrel{\text{DI}}{=} (a_1 a_2 a_3 a_4 a_5 b)^5 b \\
&= b_2 b.
\end{aligned}$$

Thus we obtain the claim.  $\square$



4. THE SECOND HOMOLOGY GROUP OF  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ 

In this section, we prove Theorem 1.2. First, we obtain a generating set for  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  when  $g \geq 9$  by using the Hopf formula and applying the discussion of Pitsch [7]. More precisely, we obtain the following proposition.

**Proposition 4.1.** *For  $g \geq 9$ ,  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  is generated by one element  $x_0$ .  $x_0$  is represented by the following element:*

$$A^{-7} B_1^{-2} B_2^{-4} B_3^{-6} B_4^4 B_5^2 B^{12} C^2,$$

where  $A, B_i$  ( $i = 1, \dots, 5$ ),  $B$  and  $C$  are the followings.

$$\begin{aligned} A &:= b^2, \\ B_i &:= a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}, \\ B &:= b a_4 b a_4^{-1} b^{-1} a_4^{-1}, \\ C &:= (a_2 a_3 a_4 a_5 a_6 b)^6 (a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} b^{-1})^6 (a_1 a_2 a_3 a_4 a_5 a_6 b)^{-4} \\ &\quad \cdot (a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} b^{-1})^{-5}. \end{aligned}$$

Now we recall the classical Hopf formula. Let  $G$  be a group with finite presentation  $G = \langle X | R \rangle$ , where  $X$  is a finite subset of  $G$  and  $R$  is a finite set consisting of words of the elements of  $X$ . Then  $G$  is isomorphic to the quotient group  $F/K$ , where  $F$  is the free group which is generated by  $X$  and  $K$  is the normal subgroup of  $F$  which is normally generated by  $R$ . The classical *Hopf formula* states that

$$H_2(G; \mathbb{Z}) \cong \frac{K \cap [F, F]}{[K, F]}.$$

We remark that  $(K \cap [F, F])/[K, F]$  is an abelian group and any element of  $(K \cap [F, F])/[K, F]$  is represented by a product of commutators of elements of  $F$  and by a product of conjugations of elements of  $R$  on  $F$ . Since  $f k f^{-1} \equiv k$  in  $K/[K, F]$  for any  $f \in F$  and  $k \in K$ , every element of  $H_2(G; \mathbb{Z})$  is represented by  $\prod r_i^{n_i}$ , where  $R = \{r_1, \dots, r_N\}$  and  $n_i \in \mathbb{Z}$ .

We modify the presentation for  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  for  $g \geq 9$  in Theorem 1.1 to apply the Hopf formula to  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  easily.

At first we easily know that  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  admits a presentation with generators  $a_0, a_1, \dots, a_{g-1}$  and relators:

- (1)  $a_i^2$  for  $i = 0, \dots, g-1$ ,
- (2)  $[a_i, a_j]$  for “ $j - i > 1$  and  $i \neq 0$ ” or “ $i = 0$  and  $j \neq 4$ ”,
- (3)  $a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}$  for  $i = 1, \dots, g-2$   
 $a_0 a_4 a_0 a_4^{-1} a_0^{-1} a_4^{-1}$ ,
- (4)  $(a_2 a_3 a_4 a_5 a_6 a_0)^6 (a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} a_0^{-1})^6 (a_1 a_2 a_3 a_4 a_5 a_6 a_0)^{-4} (a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} a_0^{-1})^{-5}$ ,
- (5)  $[a_{g-5}, b_{\frac{g-2}{2}}]$  for  $g \geq 8$  even,

where  $a_0 = b$  and  $b_{\frac{g-2}{2}}$  is inductively defined as follows:  $b_1 = a_0$ ,  $b_2 = (a_1 a_2 a_3 a_4 a_5 b)^5$ ,  $b_{i+1} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5 (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^{-6}$  for  $2 \leq i \leq \frac{g-4}{2}$ .

**Lemma 4.2.** *In the above presentation, relators  $a_1^2, \dots, a_{g-1}^2$  in (1) are unnecessary.*

*Proof.* By the relators (3), we can write  $a_1, \dots, a_{g-1}$  as conjugations of  $a_0$  in  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  inductively, as follows.

$$\begin{aligned} a_4 &= a_0 a_4 a_0^{-1} a_0^{-1}, \\ a_5 &= a_4 a_5 a_4^{-1} a_4^{-1}, & a_3 &= a_4 a_3 a_4^{-1} a_4^{-1}, \\ a_6 &= a_5 a_6 a_5^{-1} a_5^{-1}, & a_2 &= a_3 a_2 a_3^{-1} a_3^{-1}, \\ &\vdots & a_1 &= a_2 a_1 a_2^{-1} a_2^{-1}. \\ a_{g-1} &= a_{g-2} a_{g-1} a_{g-2}^{-1} a_{g-2}^{-1}, \end{aligned}$$

Thus  $a_1^2, \dots, a_{g-1}^2$  are conjugations of  $a_0^2$  in  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ . □

We set

$$\begin{aligned} A &:= a_0^2, \\ D_{i,j} &:= [a_i, a_j], \\ B_i &:= a_i a_{i+1} a_i^{-1} a_{i+1}^{-1}, \\ B &:= a_0 a_4 a_0^{-1} a_4^{-1}, \\ C &:= (a_2 a_3 a_4 a_5 a_6 a_0)^6 (a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} a_0^{-1})^6 (a_1 a_2 a_3 a_4 a_5 a_6 a_0)^{-4} \\ &\quad (a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} a_0^{-1})^{-5}, \\ D &:= [a_{g-5}, b_{\frac{g-2}{2}}]. \end{aligned}$$

Then any element  $x$  of  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  is represented by

$$x = A^n \left( \prod_{(*)} D_{i,j}^{n_{i,j}} \right) \left( \prod_{i=1}^{g-2} B_i^{m_i} \right) B^m C^l D^{l'},$$

where  $n, n_{i,j}, m_i, m, l, l' \in \mathbb{Z}$  and  $(*)$  means the condition “ $j - i > 1$  and  $i \neq 0$ ” or “ $i = 0$  and  $j \neq 4$ ”.

**Definition 4.3.** Let  $G$  and  $F$  be groups which are given in the Hopf formula. For  $g, h \in F$  such that  $[g, h] = 1$  in  $G$ , we denote by  $\{g, h\}$  the equivalence class of the commutator  $[g, h] \in [F, F]$  in  $H_2(G; \mathbb{Z})$ .

Korkmaz and Stipsicz [4, Lemma 3.3] give the following relations in  $H_2(G; \mathbb{Z})$ . For  $g, h, k \in G$  such that  $g$  commute with  $h$  and  $k$ ,

$$\begin{aligned} (I) \quad \{g, hk\} &= \{g, h\} + \{g, k\}, \\ (II) \quad \{g, h^{-1}\} &= -\{g, h\}. \end{aligned}$$

Note that relation (I) is obtained from relation (II).

Let  $\mathcal{T}(N_{g,n})$  be the subgroup of  $\mathcal{M}(N_{g,n})$  generated by all Dehn twists and  $\mathcal{M}(\Sigma_{g,n})$  the mapping class group of a compact connected orientable surface  $\Sigma_{g,n}$  of genus  $g$  with  $n$  boundary components (i.e.  $\mathcal{M}(\Sigma_{g,n})$  is the group of isotopy classes of orientation preserving self-diffeomorphisms on  $\Sigma_{g,n}$  which fix each boundary component pointwise).

**Lemma 4.4.** *Let  $g \geq 9$ . If  $\alpha$  and  $\beta$  are disjoint non-separating two-sided simple closed curves on  $N_g$  then  $\{t_\alpha, t_\beta\} = 0$  in  $H_2(\mathcal{T}(N_g); \mathbb{Z})$ , where  $t_\alpha$  and  $t_\beta$  are Dehn twists along simple closed curves  $\alpha$  and  $\beta$  respectively.*

*Proof.* Let  $S$  be the surface obtained by cutting  $N_g$  along the simple closed curve  $\alpha$  and  $g'$  the genus of  $S$ . Note that if  $g$  is even and  $S$  is orientable then  $g' = \frac{g-2}{2} \geq \frac{10-2}{2} = 4$  and if  $g$  is odd or  $S$  is non-orientable then  $g' = g - 2 \geq 7$  since  $g \geq 9$ . We regard  $t_\beta$  as an element of  $\mathcal{M}(\Sigma_{g',2})$  when  $g$  is even and  $S$  is orientable or  $\mathcal{T}(N_{g',2})$  when  $g$  is odd or  $S$  is non-orientable. Harer [2] proved that  $H_1(\mathcal{M}(\Sigma_{h,n}); \mathbb{Z}) = 1$  for  $h \geq 3$  and Stukow [9] proved that  $H_1(\mathcal{T}(N_{h,n}); \mathbb{Z}) = 1$  for  $h \geq 7$ . Thus there exist  $X_i, Y_i \in \mathcal{M}(S)$  or  $\mathcal{T}(S)$  such that  $t_\beta = \prod_i [X_i, Y_i]$ . Note that  $X_i$  and  $Y_i$  commute with  $t_\alpha$ . Therefore, in  $H_2(\mathcal{T}(N_g); \mathbb{Z})$ , we have

$$\begin{aligned} \{t_\alpha, t_\beta\} &= \left\{ t_\alpha, \prod_i [X_i, Y_i] \right\} \stackrel{(I)}{=} \sum_i \{t_\alpha, [X_i, Y_i]\} \\ &\stackrel{(I) \& (II)}{=} \sum_i \left[ \{t_\alpha, X_i\} + \{t_\alpha, Y_i\} - \{t_\alpha, X_i\} - \{t_\alpha, Y_i\} \right] \\ &= 0. \end{aligned}$$

Thus we obtain the claim.  $\square$

The homomorphism  $\rho_2|_{\mathcal{T}(N_g)} : \mathcal{T}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  induces a homomorphism  $H_2(\mathcal{T}(N_g); \mathbb{Z}) \rightarrow H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  on their homology groups. Hence the equivalence classes of  $D_{i,j}$  and  $D$  in  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  are trivial by Lemma 4.4 and any element of  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  is represented by

$$x = A^n \left( \prod_{i=1}^{g-2} B_i^{m_i} \right) B^m C^l.$$

**Proof of Proposition 4.1.** By the Hopf formula, any element of  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  is a product of commutators of the free group generated by  $\{a_0, a_1, \dots, a_{g-1}\}$ . Hence the exponent sum of each  $a_i$  in  $x$  is zero. The exponent sum of each  $a_i$  in  $x$  is the following.

$$\begin{aligned} (\text{the exponent sum of } a_0) &= 2n + m + l, \\ (\text{the exponent sum of } a_1) &= m_1 + l, \\ (\text{the exponent sum of } a_2) &= -m_1 + m_2 + l, \\ (\text{the exponent sum of } a_3) &= -m_2 + m_3 + l, \\ (\text{the exponent sum of } a_4) &= -m_3 + m_4 - m + l, \\ (\text{the exponent sum of } a_5) &= -m_4 + m_5 + l, \\ (\text{the exponent sum of } a_6) &= -m_5 + m_6 + l, \\ (\text{the exponent sum of } a_7) &= -m_6 + m_7, \\ &\vdots \\ (\text{the exponent sum of } a_{g-2}) &= -m_{g-3} + m_{g-2}, \\ (\text{the exponent sum of } a_{g-1}) &= -m_{g-2}. \end{aligned}$$

The above equations give  $m_{g-2} = m_{g-3} = \cdots = m_7 = m_6 = 0$  and the following system of equations.

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By an elementary calculation, this matrix has rank 7 and so the linear map  $\mathbb{Z}^8 \rightarrow \mathbb{Z}^7$  has a 1-dimensional kernel. We can check the kernel is generated by the vector  $(-7, -2, -4, -6, 4, 2, 12, 2)$ . Therefore  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  is generated by  $x_0$  which is represented by an element

$$A^{-7} B_1^{-2} B_2^{-4} B_3^{-6} B_4^4 B_5^2 B^{12} C^2.$$

Thus we finish the proof.  $\square$

When  $g \geq 7$  is odd, Theorem 1.2 is proved by Stein [8]. It is sufficient for a proof of Theorem 1.2 to show that  $x_0 = 0$  when  $g \geq 10$  is even.

**Proof of Theorem 1.2.** Recall that  $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  is isomorphic to  $O(g, \mathbb{Z}_2) = \{A \in GL(g, \mathbb{Z}_2) \mid {}^t A A = E\}$ . Under this identification, we define the injective homomorphism

$$\begin{aligned} \iota_g : \text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot) &\hookrightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot) \\ A &\mapsto \left( \begin{array}{c|c} A & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline 0 & \dots & 0 & 1 \end{array} \right). \end{aligned}$$

Note that  $\iota_g(a_i) = a_i$  for  $i = 1, \dots, g-2$  and  $\iota_g(b) = b$ . Let  $F$  and  $F'$  be free groups generated by  $\{a_1, \dots, a_{g-1}, b\}$  and  $\{a_1, \dots, a_{g-2}, b\}$  respectively and  $\nu : F \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$  and  $\nu' : F' \rightarrow \text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot)$  natural projections. Then the following diagram is commutative.

$$\begin{array}{ccc} F' & \xrightarrow{\tilde{\iota}_g} & F \\ \nu' \downarrow & \circlearrowleft & \downarrow \nu \\ \text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot) & \xrightarrow{\iota_g} & \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot) \end{array}$$

The homomorphism  $\tilde{\iota}_g : F' \rightarrow F$  is defined by  $\tilde{\iota}_g(a_i) = a_i$  for  $i = 1, \dots, g-2$  and  $\tilde{\iota}_g(b) = b$ . We denote the kernels of  $\nu$  and  $\nu'$  by  $K$  and  $K'$  respectively. By the Hopf formula, the restriction  $\tilde{\iota}_g : K' \cap [F', F'] \rightarrow K \cap [F, F]$  of  $\tilde{\iota}_g$  induces the homomorphism  $\tilde{\iota}_{g*} : H_2(\text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot); \mathbb{Z}) \rightarrow H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ . Since  $H_2(\text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot); \mathbb{Z}) = 0$  for  $g \geq 10$  even ([8]), it is enough for the proof of Theorem 1.2 to show that  $\tilde{\iota}_{g*}$  is surjective for  $g \geq 10$ . By Proposition 4.1,  $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$  is generated by  $x_0$  for  $g \geq 9$  such that  $x_0$  is represented

by  $A^{-7}B_1^{-2}B_2^{-4}B_3^{-6}B_4^4B_5^2B^{12}C^2$ . Thus we can check  $\tilde{\iota}_{g*}(x'_0) = x_0$  by the definition of  $\tilde{\iota}_g$ , where  $x'_0$  is represented by an element  $A^{-7}B_1^{-2}B_2^{-4}B_3^{-6}B_4^4B_5^2B^{12}C^2 \in K' \cap [F', F']$ . Therefore  $x_0$  is trivial and we complete the proof.  $\square$

**Acknowledgement:** The authors would like to express his gratitude to Hisaaki Endo, for his encouragement and helpful advices. The authors also wish to thank Susumu Hirose for his comments and helpful advices.

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(RYOMA KOBAYASHI) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY,  
TOKYO UNIVERSITY OF SCIENCE, NODA, CHIBA, 278-8510, JAPAN  
E-mail address: [kobayashi\\_ryoma@ishikawa-nct.ac.jp](mailto:kobayashi_ryoma@ishikawa-nct.ac.jp)

(GENKI OMORI) DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, OH-  
OKAYAMA, MEGURO, TOKYO 152-8551, JAPAN  
E-mail address: [omori.g.aa@m.titech.ac.jp](mailto:omori.g.aa@m.titech.ac.jp)